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A MODIFIED MICKENS PROCEDURE FOR CERTAIN NON-LINEAR OSCILLATORS

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Consider a non-linear oscillator modelled by the equation

$$x'' + f(x) = 0, \quad x(0) = A, \; x'(0) = 0, \tag{1}$$

where A is a given positive constant and $f(x)$ satisfies the condition

$$f(-x) = -f(x) \tag{2}$$

and its derivative near $x = 0$ is non-negative. The system will oscillate between symmetric limits $[-A, A]$. Many techniques exist for constructing analytical approximations to the solution to the oscillatory system: the Lindstedt–Poincaré method [1, 2], multi-time expansion [1, 2], harmonic balancing [1, 2], the averaging technique [1–3], and the iteration procedure [4]. These methods, except the harmonic balancing, apply to weakly non-linear cases only. The method of harmonic balance is capable of producing first analytical approximation to the solution to the non-linear system, valid even for rather large values of oscillation amplitude. But it is usually rather difficult to apply the method to produce higher-order analytical approximations to the solution because they require solving sets of equations with very complex non-linearity. Recently, by combining the linearization of governing equation with the method of harmonic balance, Wu, Lim and their associates [5–7] have established approximate analytical periods for the non-linear system, which is valid for a wide range of values of oscillation amplitude.

The purpose of this letter is to generalize the Mickens' iteration procedure such that excellent approximate analytical solutions, valid for small as well as large values of oscillation amplitude, can be determined for equation (1). We will use the Duffing equation, as an example, to illustrate the applicability and accuracy of the method.

To proceed, one rewrites equation (1) to read [4]

$$x'' + \omega^2 x = \omega^2 x - f(x) := g(x), \tag{3}$$

where ω is *a priori* unknown frequency of the periodic solution $x(t)$ being sought. The proposed iteration scheme is

$$x''_{k+1} + \omega^2 x_{k+1} = g(x_{k-1}) + g_x(x_{k-1})(x_k - x_{k-1}), \quad k = 0, 1, 2, \ldots, \tag{4}$$

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where the inputs of starting functions are

$$x_{-1}(t) = x_0(t) = A\cos\omega t. \tag{5}$$

It is further required that for each k, the solution to equation (5), is to satisfy the initial conditions

$$x_k(0) = A, \qquad x_k'(0) = 0, \; k = 1, 2, 3, \ldots. \tag{6}$$

Note that, for given $x_{k-1}(t)$ and $x_k(t)$, equation (4) is a second order, inhomogeneous differential equation for $x_{k+1}(t)$. Its right side can be expanded into the following Fourier series:

$$g[x_{k-1}(t)] + g_x[x_{k-1}(t)][x_k(t) - x_{k-1}(t)] = a_1(A,\omega)\cos\omega t + \sum_{n=2}^{N} a_{2n-1}(A,\omega)\cos[(2n-1)\omega t], \tag{7}$$

where the coefficients $a_{2n-1}(A,\omega)$ are known functions of A and ω, and the integer N depends upon the function $g(x)$ on the right-hand side of equation (3). In view of equation (7), the solution to equation (4) is taken to be

$$x_{k+1}(t) = B\cos\omega t - \sum_{n=2}^{N} \frac{a_{2n-1}(A,\ \omega)}{[(2n-1)^2 - 1]\omega^2}\cos[(2n-1)\omega t], \tag{8}$$

where B is, tentatively, an arbitrary constant. In equation (8), the particular solution is chosen such that it contains no secular terms [1, 2, 4], which requires that the coefficient $a_1(A,\omega)$ of right-side term $\cos\omega t$ in equation (7) satisfy

$$a_1(A,\omega) = 0. \tag{9}$$

Equation (9) allows the determination of the frequency ω as a function of A. Next, the unknown constant B will be computed by imposing the initial conditions in equation (6). Finally, putting these steps together gives the solution $x_{k+1}(t)$.

This procedure can be performed to any desired iteration step k. However, for most problems calculations can be stopped at $k = 2$. As we will show in the following example, termination at $k = 2$ is capable of providing excellent approximate analytical representations to the exact solution, valid for small as well as large values of oscillation amplitude.

The use of this iteration procedure may be illustrated by the following Duffing equation:

$$x'' + x + \varepsilon x^3 = 0, \quad x(0) = A, \; x'(0) = 0. \tag{10}$$

The first iteration of the starting functions shown in equation (5) leads to

$$x_1'' + \omega^2 x_1 = \left(\omega^2 - 1 - \frac{3\varepsilon A^2}{4}\right)A\cos\omega t - \frac{\varepsilon A^3}{4}\cos(3\omega t). \tag{11}$$

The requirement of no secular terms in $x_1(t)$ implies that

$$\omega = \omega_1 = \sqrt{1 + \frac{3\varepsilon A^2}{4}}. \tag{12}$$

Equation (12) gives the first approximate frequency of equation (1) and the corresponding approximate periodic solution is

$$x_1(t) = A\cos\omega t + \left(\frac{\varepsilon A^3}{32\omega^2}\right)(-\cos\omega t + \cos 3\omega t), \tag{13}$$

where the frequency ω is listed in equation (12).

Using equations (4), (5) and (13) yields the following equation for determining $x_2(t)$ from the second iteration:

$$x_2'' + \omega^2 x_2 = \left[(\omega^2 - 1)\left(A - \frac{\varepsilon A^3}{32\omega^2}\right) - \frac{3\varepsilon A^3}{4} + \frac{3\varepsilon^2 A^5}{64\omega^2}\right]\cos\omega \mathrm{t} - \left[\frac{7\varepsilon A^3}{32} + \frac{4\varepsilon A^3 + 3\varepsilon^2 A^5}{128\omega^2}\right]\cos 3\omega t$$
$$- \left(\frac{3\varepsilon^2 A^5}{128\omega^2}\right)\cos 5\omega t. \tag{14}$$

The condition that there be no secular terms in the solution $x_2(t)$ requires that

$$\omega = \omega_2 = \frac{\sqrt{32 + 25\varepsilon A^2 + \sqrt{1024 + 1472\varepsilon A^2 + 433\varepsilon^2 A^4}}}{8}. \tag{15}$$

Equation (15) expresses the second approximate frequency of equation (1) and the corresponding approximate periodic solution is given by

$$x_2(t) = A\cos\omega \mathrm{t} - \left[\frac{7\varepsilon A^3}{256\omega^2} + \frac{4\varepsilon A^3 + 3\varepsilon^2 A^5}{1024\omega^4}\right](\cos\omega t - \cos 3\omega t)$$
$$- \left[\frac{\varepsilon^2 A^5}{1024\omega^4}\right](\cos\omega t - \cos 5\omega t), \tag{16}$$

where ω is given in equation (15).

The exact frequency of the periodic motion of the Duffing equation is given by [1]

$$\omega = \omega_e = \frac{\pi\sqrt{1 + \varepsilon A^2}}{2}\left(\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{-m\sin^2}\theta}\right)^{-1}, \quad m = \frac{\varepsilon A^2}{2(1 + \varepsilon A^2)} \tag{17}$$

and the corresponding approximate frequency obtained by the Mickens' iteration procedure [4] is

$$\omega_p = 1 + \frac{3\varepsilon A^2}{8} - \frac{21\varepsilon^2 A^4}{256}. \tag{18}$$

For comparison, the exact frequency ω_e obtained by integrating equation (17) and the approximate frequencies ω_p, ω_1 and ω_2 computed by equations (18), (12) and (15), respectively, are listed in Table 1. Table 1 indicates that the original Mickens' procedure

TABLE 1

Comparison of approximate frequencies with the corresponding exact frequency for the Duffing equation

εA^2	ω_e Equation (17) [1]	ω_p Equation (18), [4]	ω_1 Equation (12), present	ω_2 Equation (15), present
0·2	1·07200	1·07172	1·07238	1·07200
0·4	1·13891	1·13687	1·14018	1·13889
0·6	1·20173	1·19547	1·20416	1·20170
0·8	1·26118	1·24750	1·26491	1·26112
1	1·31778	1·29297	1·32288	1·31767
2	1·56911	1·42188	1·58114	1·56873
5	2·15042	0·82422	2·17945	2·14912
10	2·86664	†	2·91548	2·86408
100	8·53359	†	8·71780	8·52220
1000	26·8107	†	27·4044	26·7734
10000	84·7245	†	86·6083	84·6088

† Meaningless results ($\omega_p < 0$)

fails to yield practical solutions to the Duffing equation for large values of εA^2. We also have

$$\lim_{\varepsilon A^2 \to +\infty} \frac{\omega_1}{\omega_e} = \frac{\sqrt{3}}{\pi} \int_0^{\pi/2} \frac{1}{\sqrt{1 - 0.5\ \sin^2\theta}}\, d\theta = 1{\cdot}0222, \tag{19}$$

$$\lim_{\varepsilon A^2 \to +\infty} \frac{\omega_2}{\omega_e} = \frac{\sqrt{25 + \sqrt{433}}}{4\pi} \int_0^{\pi/2} \frac{1}{\sqrt{1 - 0.5\ \sin^2\theta}}\, d\theta = 0{\cdot}998596. \tag{20}$$

These formulas show that formula (15) is more accurate than formula (12), and can give excellent approximate frequencies for both small and large values of oscillation amplitude.

In summary, a modified Mickens' iteration procedure has been proposed to solve non-linear oscillations of single-degree-of-freedom systems with odd non-linearity. While many possible iteration schemes can be formulated for solving equation (1), the one proposed in this paper is of the feature of providing excellent approximate analytical solutions, valid for small as well as large values of oscillation amplitude. In fact, this modified iteration procedure not only derives more accurate analytical solutions, but also extends the validity of the original Mickens' procedure to a larger parameter regime where the original procedure fails. The details of the method have been illustrated by a worked example. For instance, disregard how large the parameters are, the discrepancies of the approximate analytical solutions to the Duffing equation with respect to the exact solution for the first and second iteration never exceed 2·22% and 0·14% as indicated in equations (19) and (20), respectively. The iteration procedure can be carried on if solutions of higher degree of accuracy are required.

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